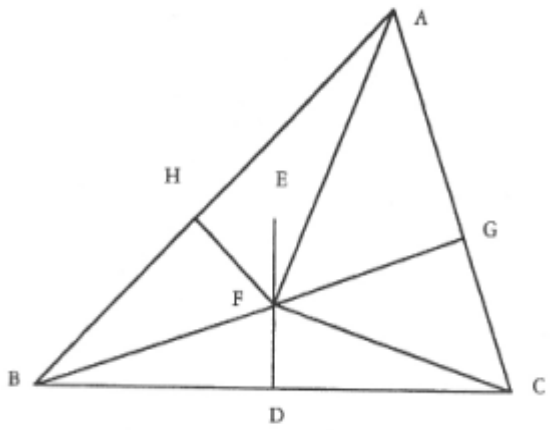
All triangles are isosceles...

The English author Lewis Carroll (1832 –1898) found fame after the publication of his book *Alice's Adventures in Wonderland*. But less well known, is the "other" Lewis Carroll, the mathematician at Christ Church College, Oxford. Within the academic discipline of mathematics, he worked primarily in the fields of recreational mathematics, producing a dozen books. In one of these, he developped the following proof that every triangle has two equal sides :



Let ABC be any triangle. Bisect BC at D and from D draw DE at right angles to BC. Draw line AF that bisects ∠BAC. Join FB, FC, and from F, draw FG, FH, at right angles to AC, AB.

(1) ∠FGA = ∠FHA;

(2) ∠FAG = ∠FAH;

(3) Triangles AFG and AFH have the side AF in common;

(4) Therefore, AG = AH.

(5) Similarly, we have FG = FH.

(6) FB = FC.

(7) Therefore, combining steps (5) and (6), we have GC = HB.

(8) Combining steps (4) and (7), AC = AB.

Therefore, triangle ABC is isosceles.

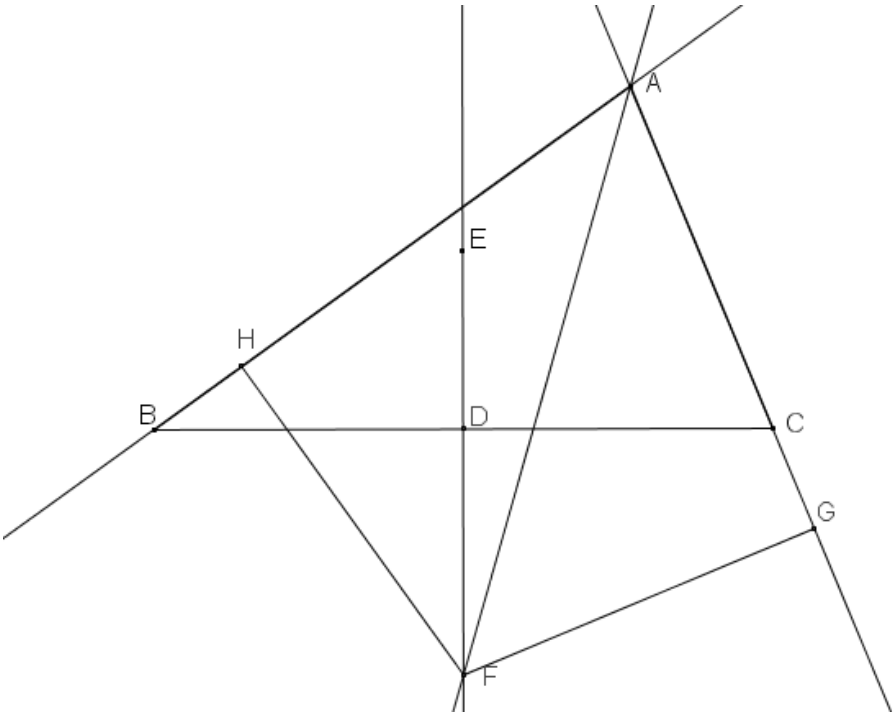
Adapted from Rediscovered Lewis Carrol puzzles by Edward Wakeling

Tasks

1. Propose a short summary of the subject and give your first impression.
2. Line DE is known as a remarkable line in triangle ABC : give some more information about such a line.
3. Line AF is known as a remarkable line in triangle ABC : give some more information about such a line.
4. Resume step by step Caroll's demonstration and justify each statement.
5. By drawing your own picture accurately, find out the fallacy in Caroll's proof.
6. Talk about any other famous English mathematicians you've heard of.

All triangles are isosceles...

A donner éventuellement au candidat lorsqu'il aborde la question 5

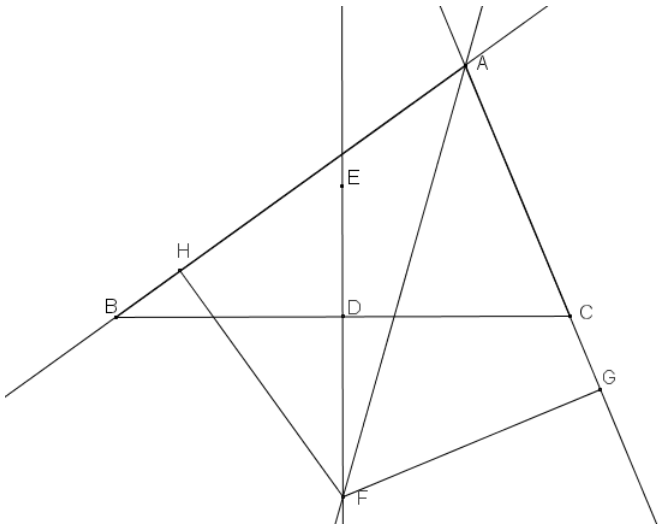


All triangles are isosceles...

Comments and answers

1. The document is proposing a geometrical proof of a false situation given by Lewis Carrol.
2. Line DE is the perpendicular bisector of line segment BC. It's the locus of points in the plane that are the same distance from points B and C.The three perpendicular bisectors of the sides of a triangle are concurrhgent and intersect at a point called the circumcentre, or the centre of the the circumscribing circle of the triangle.
3. Line AF is the angle bisector of angle BAC. It's the locus of points in the plane that are the same distance from lines AB and AC. The three angle bisectors of the internal angles of a triangle are concurrent and intersect at a point called the incentre, or the centre of the incircle of the triangle, the circle that is tangent to the triangle.
4. Step by step justifications:
   (1) $\angle$FGA = $\angle$FHA because they're both right angles by construction of points G and H;
   (2) $\angle$FAG = $\angle$FAH because line AF bisects angle BAC;
   (3) Triangles AFG and AFH have the side AF in common which is obvious !;
   (4) In right triangle AFG with right angle at point G, AG = AF $\times \cos \widehat{FAG}$, and in right triangle AFH with right angle at point H, AH = AF $\times \cos \widehat{FAH}$. Thus, using (2), AG = AH;
   (5) Similarly, we have FG = FH since FG = AF $\times \sin \widehat{FAG}$ and FH = AF $\times \sin \widehat{FAH}$;
   (6) FB = FC since F belongs to the perpendicular bisector of line segment BC;
   (7) Using the Pythagoras theorem in right triangles FGC, FHB, with right angle at point G, H, we have : $GC^2 = FC^2 - FG^2$ and $HB^2 = FB^2 - FH^2$, leading to $GC^2 = HB^2$ from (5) and (6), and therefore, GC = HB;
   (8) Combining steps (4) and (7), AC = AG + GC = AH + HB = AB;

   Therefore, triangle ABC is isosceles since it has two equal sides.
5. An accurate drawing shows that point F is outside the triangle… Thus, one of the points G, H is also outside the triangle, so that the statement AC = AG + GC and AB = AH + HB is false : on the following picture, we still have AB = AH + HB but AC = AG – GC, and not AG + GC !



6. Newton and his works concerning the derivative of a function in the 17th century.